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LETTER TO THE EDITOR

Orbiting and surface waves

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Abstract. In the framework of a semiclassical approximation, the singular trajectories—as well as the analytical ones—are considered. As a result one exhibits the close analogy between orbiting and surface waves.

It has been shown (Knoll and Schaeffer 1976) that in the limit of $\hbar \to 0$ the scattering amplitude can be written as a sum of terms, each of them being associated with a trajectory. However, until now, this procedure has not been applied to the case of a purely real potential, due to the orbiting singularity. The purpose of this letter is to outline a method for extending the semiclassical theory to this case; in this way we single out, on the one hand, 'classical' effects like rainbow and orbiting (Ford and Wheeler 1959), and on the other hand, typical diffractive effects like surface waves (Di Salvo and Viano 1982, 1984, Keller 1958, Levy and Keller 1959, Nussenzveig 1969). We refer to a central (real) nuclear potential, such that the radial component of the momentum has three and only three turning points, r_1, r_2 and r_3, according to the convention adopted by Brink and Takigawa (1977); however our treatment could be extended to other cases, as we shall see.

We begin by analysing the trajectories which give a contribution to the scattering process. Besides the analytical (real and complex) trajectories (Knoll and Schaeffer 1976), one has to take into account the singular trajectories (Berry and Mount 1972). These last are particularly important when a classical trajectory grazes a surface where the potential varies rapidly; in our case, since orbiting—i.e. the logarithmic divergence of the deflection function—is produced just by the rapid variation of the nuclear potential, we shall take into account the singular trajectories propagating along the surface of the sphere with a radius equal to the orbiting radius r_{12} (such a surface shall be called, from now on, Ξ).

For later convenience we define λ as the classical angular momentum and we call λ_{12} the orbiting angular momentum.

Let us consider, firstly, the analytical trajectories. Among the real trajectories, those with $\lambda > \lambda_{12}$ are either of 'Coulomb' type or of 'nuclear' type (Knoll and Schaeffer 1976), whereas those with $\lambda < \lambda_{12}$ penetrate inside Ξ and get out of it without any internal reflections; the remaining trajectories are complex and describe the propagation in the dark side of the rainbow (Berry 1966), the direct reflection outside Ξ and the (single or multiple) reflection inside Ξ; in particular the trajectories with $\mathrm{Re}\,\lambda > \lambda_{12}$ may penetrate inside Ξ by the tunnel effect across the barrier of the effective potential. Now multiple reflection causes resonances at some particular energies, such that the angular momentum is as close as possible to a half-integer value; in this case the internal reflection coefficient has a relative maximum. Moreover the logarithmic divergence of the WKB deflection function at λ_{12} is smoothed off by barrier penetration effects, so as to have a minimum, say $-\vartheta_0$, close to λ_{12} (Ford and Wheeler 1959); therefore the 'classical' trajectory with angular momentum λ_{12} has to be intended to be modified by these corrective terms; we shall return to this point later on.

As regards the singular trajectories, these behave like the diffracted rays in optics (Keller 1958): they consist of two branches of the 'classical' trajectory of angular momentum λ_{12}, joined smoothly by an arc of geodesics on Ξ; moreover the surface ray may generate one or more 'shortcuts', i.e. branches of classical trajectory which penetrate inside Ξ and are tangential to this surface. Other singular trajectories are generated by the complex trajectory of angular momentum λ_{23} such that the turning points r_2 and r_3 coincide, since this trajectory also has a divergent deflection angle. These surface waves are worth being mentioned, despite their negligible contribution to the scattering amplitude, since they travel from the 'shadow' region to the 'lit' region, similarly to those which were exhibited mathematically by Nussenzveig (1969) in the case of a sharp-edged sphere.

The probability amplitude to be associated with each trajectory may be determined by applying the Hamilton-Jacobi equation and the transport equation (Knoll and Schaeffer 1976, Di Salvo and Viano 1982, 1984) to each analytical trajectory (or branch of analytical trajectory). As regards the singular trajectories, one has to apply the rules stated by Levy and Keller (1959) (see also Di Salvo and Viano 1982, 1984, Nussenzveig 1969) for the diffraction by a smooth object; at this stage some proportionality coefficients—like decay exponents, diffraction coefficients, limiting refraction or limiting (internal) reflection coefficients—are left undetermined. Below, we are going to outline a method which allows us to find a relationship between these coefficients and the nuclear potential; detailed proofs and generalisations are left to a forthcoming paper.

To this end we recall the approximate scattering amplitude written by Brink and Takigawa (1977) for a nuclear plus Coulomb potential. Using the Poisson sum formula and taking the Debye expansion of the S function (Anni and Renna 1981), it can be shown that the scattering amplitude can be written as

$$f(k,\vartheta)=\sum_{p=0}^{\infty}\sum_{m=-\infty}^{\infty}[f_m^{p+}(k,\vartheta)+f_m^{p-}(k,\vartheta)] \qquad (1)$$

where

$$f_m^{p\pm}(k,\vartheta)=\frac{(-1)^{m+p+1}}{\mathrm{i}k(2\pi\sin\vartheta)^{1/2}}\int_0^\infty \lambda^{1/2}\exp\{\mathrm{i}[2\eta_p(\lambda)+2m\pi\lambda\pm(\lambda\vartheta-\tfrac{1}{4}\pi)]\}\,\mathrm{d}\lambda \qquad (2)$$

and

$$\eta_p(\lambda)=\delta_1+\mathrm{i}\pi\varepsilon(1-\delta_{p,0})+p\delta_{32}(\lambda)+\tfrac{1}{2}(p+1)\log[N(\mathrm{i}\varepsilon)] \qquad (2a)$$

δ_1 is the WKB phase shift relative to the trajectory of angular momentum λ which has only one turning point, at $r=r_1$; the remaining functions, i.e. $S_{32}(\lambda)$, ε and $N(\mathrm{i}\varepsilon)$, are defined in the paper by Brink and Takigawa (1977). In an asymptotic evaluation for $\hbar\to 0$ of the integrals (2) one has to determine the locations of the saddle points of the phases of such integrals. These are given by the roots of the equation

$$\Theta^p(\lambda)=\pm\vartheta-2m\pi \qquad (3)$$

where $\Theta^p(\lambda) = 2\,\mathrm{d}\eta_p/\mathrm{d}\lambda$ is the deflection function, which results in it being smoothed off at $\lambda = \lambda_{12}$, the singularities of $2\eta_p$ and $2S_{32}$ being compensated exactly by the singularity of $\mathrm{i}\log[N(\mathrm{i}\varepsilon)]$ at the orbiting angular momentum. Therefore the real part of each deflection function presents a minimum, $-\vartheta_p = -(\vartheta_0 + p\vartheta_t)$, close to λ_{12}, ϑ_t being the amplitude of the arc corresponding to the shortcut.

Now if the deflection angle, ϑ', is greater than $-\vartheta_p$, it can be shown that the asymptotic evaluation of each integral (2) is conveniently made by deforming the path of integration in order to coincide with the steepest descent paths of the saddle points.

On the other hand, if $\vartheta' < -\vartheta_p$, one has to consider the contribution of the residues at the poles of the integrand; these last are of pth order and fulfil the equation

$$\varepsilon(\lambda) = \mathrm{i}(n + \tfrac{1}{2}) \qquad n \geqslant 0. \tag{4}$$

This equation indicates that the poles lie in the first quadrant of the complex λ plane, on a line which crosses the real axis perpendicularly at $\lambda = \lambda_{12}$.

Therefore the asymptotic evaluation of the scattering amplitude (1) consists of two sums, one over the residues at the poles and the other over the saddle points. Comparison with the approximate scattering amplitude suggested above indicates that the saddle point contributions correspond to the analytical trajectories, whereas the residues at the poles correspond to the singular trajectories; in any case the pth term represents a trajectory which has p branches inside Ξ. Such a comparison allows us also to determine the above-mentioned proportionality coefficients once the potential is given; of course these coefficients may be determined numerically. However for n sufficiently small and $r_{12}(\mu|V_B''|)^{1/2}(2\hbar)^{-1} \gg 1$ (a condition which is frequently met in those cases of interest) one can give some approximate analytical formulae. For the decay exponents we have

$$\alpha_n = (n + \tfrac{1}{2}) r_{12}^2 (\mu|V_B''|)^{1/2}/\hbar\lambda_{12} \tag{5}$$

where μ is the mass of the projectile and V_B'' is the value of the second derivative of the effective potential at the barrier top. Similarly for the diffraction coefficients we get

$$D_n \simeq \mathrm{i}^n \left(\frac{\mu}{|V_B''|}\right)^{1/4} \frac{r_{12}(\cos\vartheta_0/2)^{1/4}}{\mathrm{e}^{\mathrm{i}\pi/4}(\hbar^2\lambda_{12})^{1/4}\sqrt{n!}}. \tag{6}$$

Lastly the limiting refraction coefficients and the direct internal limiting reflection coefficients are given, respectively, by

$$D_{12}D_{21} \simeq \frac{(-1)^n r_{12}^2(\mu|V_B''|)^{1/2}}{n!(2\pi)^{1/2}\mathrm{i}\hbar\lambda_{12}} \tag{7}$$

and

$$R_{22} \simeq -3\mathrm{i}/(2\pi)^{1/2}(\mathrm{d}/\mathrm{d}z)[(z+n)\Gamma(z)]_{z=-n}. \tag{8}$$

Let us conclude this letter with some observations.

(i) The surface wave contribution is very similar to the orbiting amplitude, since both of them are exponentially damped; moreover the decay exponent corresponding to $n = 0$ in equation (5) coincides with the one determined by Ford and Wheeler (1959) relative to the trajectories with angular momentum slightly greater than λ_{12}.

(ii) The present treatment could be extended to the case when the potential has an absorptive part; in this case the number of terms of the Debye expansion to be considered is reduced, according to the intensity of the absorption.

(iii) The surface wave contribution, together with the shortcut mechanism, is very important in the explanation of the anomalous large angle scattering (ALAS) in nuclear physics (Di Salvo and Viano 1982, 1984, Di Salvo 1983). Moreover in the semiclassical treatment of Brink and Takigawa (1977) two Debye terms, corresponding to $p = 0$ and $p = 1$, are needed for explaining the features of α-^{40}Ca elastic scattering at $E = 29$ MeV.

(iv) Lastly the treatment presented now could be generalised to other potentials, like the atomic ones; also in the case of local, non-spherical potentials an extension could be made, thanks to the localisation principle (Levy and Keller 1959).

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